

# NUMERICAL OPTIMIZATION OF SANDWICH PLATES

## ВЫЧИСЛИТЕЛЬНАЯ ОПТИМИЗАЦИЯ МНОГОСЛОЙНЫХ ПЛИТ

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**Abstract:** The paper deals with a numerical approach of modelling of sandwich plates and numerical optimization. Finding the optimal design of sandwich thickness is a three-step process. At first the design was formulated in terms specific to the problem. Secondly, the problem was recasted as a mathematical optimization problem. Finally, the problem was solved using available optimization methods. For solved illustrative example are depicted distributions of numerical results during optimization process.

**KEYWORDS:** SANDWICH 2-D STRUCTURES, MATHEMATICAL OPTIMIZATION FORMULATION, FUNCTION APPROXIMATION, THE MODIFIED FEASIBLE DIRECTION METHOD, THE SEQUENTIAL LINEAR PROGRAMMING METHOD

### 1. Introduction

Sandwich plates are three-layer lightweight structures consisting of a soft core covered by stiff skin layers which can be made of different materials, like thin metal plates, laminated or fibre reinforced composites. Sandwich structures are characterized by both an excellent bending stiffness and low weight [7].

The assumptions for macro-mechanical modeling of sandwiches are [2]:

1. The thickness of the core is much greater than thickness of the skins:  
$$h_2 \gg h_1, h_3$$
2. The vector of strain is linear through the core:  
$$\boldsymbol{\varepsilon}(x, y, z) = \bar{\boldsymbol{\varepsilon}}(x, y) + z \boldsymbol{\kappa}(x, y)$$
$$-\frac{h_2}{2} \leq z \leq +\frac{h_2}{2}$$
3. The sheets only transmit stresses  $\sigma_x$ ,  $\sigma_y$ ,  $\tau_{xy}$  and the in-plane strains are uniform through the thickness of the skins. The transverse shear stresses  $\tau_{xz}$ ,  $\tau_{yz}$  are neglected within the skin.
4. The core transmits only transverse shear stresses, the stresses  $\sigma_x$ ,  $\sigma_y$ ,  $\tau_{xy}$  are neglected.
5. The strain  $\varepsilon_z$  is neglected in the sheets and the core.

### 2. Modelling of sandwich plates

#### 2.1. Strain relations

The strains relations are [1]:

$$\varepsilon_x = \frac{\partial u}{\partial x} = \frac{\partial \bar{u}}{\partial x} - z \frac{\partial \psi_x}{\partial x} \quad \varepsilon_y = \frac{\partial v}{\partial y} = \frac{\partial \bar{v}}{\partial y} - z \frac{\partial \psi_y}{\partial y}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x} - z \left( \frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} \right)$$

$$\gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = \frac{\partial \bar{w}}{\partial x} - \psi_x$$

$$\gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = \frac{\partial \bar{w}}{\partial y} - \psi_y$$

#### 2.2. Stress resultants and stress analysis

The in-plane resultants  $\mathbf{N}$  for sandwiches are defined by:

$$\mathbf{N} = \int_{-\frac{1}{2}h_2}^{\frac{1}{2}h_2} \boldsymbol{\sigma} dz + \int_{\frac{1}{2}h_2}^{\frac{1}{2}h_2+h_3} \boldsymbol{\sigma} dz$$

The moment resultants are given by:

$$\mathbf{M} = \int_{-\frac{1}{2}h_2}^{\frac{1}{2}h_2} z \boldsymbol{\sigma} dz + \int_{\frac{1}{2}h_2}^{\frac{1}{2}h_2+h_3} z \boldsymbol{\sigma} dz$$

and the transverse shear force by:

$$\mathbf{V} = \int_{-\frac{1}{2}h_2}^{\frac{1}{2}h_2} \boldsymbol{\tau} dz$$

The constitutive equations can be written in the condensed hypermatrix form:

$$\begin{pmatrix} \mathbf{N} \\ \mathbf{M} \\ \mathbf{V} \end{pmatrix} = \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{0} \\ \mathbf{C} & \mathbf{D} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \bar{\mathbf{A}} \end{pmatrix} \begin{pmatrix} \bar{\boldsymbol{\varepsilon}} \\ \boldsymbol{\kappa} \\ \boldsymbol{\gamma} \end{pmatrix}$$

with the stiffness coefficients

$$A_{ij} = A_{ij}^{(1)} + A_{ij}^{(3)} \quad B_{ij} = \frac{1}{2} h_2 (A_{ij}^{(3)} - A_{ij}^{(1)})$$

$$C_{ij} = C_{ij}^{(1)} + C_{ij}^{(3)}, \quad D_{ij} = \frac{1}{2} h_2 (C_{ij}^{(3)} - C_{ij}^{(1)})$$

and

$$A_{ij}^{(1)} = \int_{-\frac{1}{2}h_2}^{\frac{1}{2}h_2} E_{ij} dz = \sum_{k=1}^{n_1} E_{ij}^k h \quad A_{ij}^{(3)} = \int_{\frac{1}{2}h_2}^{\frac{1}{2}h_2+h_3} E_{ij} dz = \sum_{k=1}^{n_3} E_{ij}^k h$$

$$C_{ij}^{(1)} = \int_{-\frac{1}{2}h_2}^{\frac{1}{2}h_2} E_{ij} z dz = \sum_{k=1}^{n_1} E_{ij}^k h^k \bar{z}$$

$$C_{ij}^{(3)} = \int_{\frac{1}{2}h_2}^{\frac{1}{2}h_2+h_3} E_{ij} z dz = \sum_{k=1}^{n_3} E_{ij}^k h^k \bar{z}$$

$i, j = 1, 2, 3$

and

$$\bar{A}_{ij} = E_{ij}^t h_2 \quad i, j = 5, 6$$

$n_1$  and  $n_2$  are the numbers of layers in the lower and the upper sheet respectively and  $E_{ij}^t$  are the transverse shear moduli of the core.

### 2.3. Equilibrium equations

The equilibrium equations are formulated for a plate element (Fig. 1) and yield three force equations:

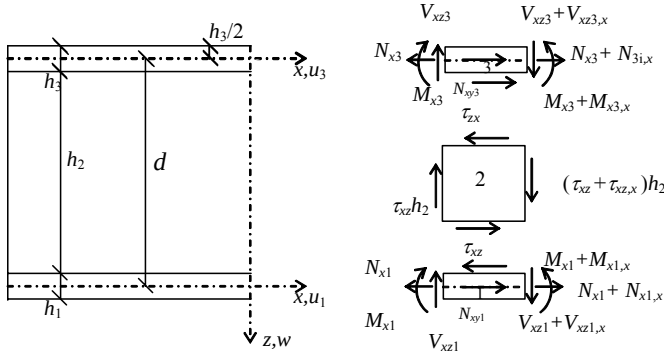


Fig. 1: Stress resultants applied to a plate element

$$\begin{aligned} \frac{\partial N_{xi}}{\partial x} + \frac{\partial N_{yi}}{\partial y} + \frac{\partial V_{zi}}{\partial z} &= 0 \\ \frac{\partial N_{xyi}}{\partial x} + \frac{\partial N_{yi}}{\partial y} + \frac{\partial V_{zyi}}{\partial z} &= 0 \quad i=1,3 \\ \frac{\partial V_{xz}}{\partial x} + \frac{\partial V_{yz}}{\partial y} + p &= 0 \end{aligned}$$

with:

$$\begin{aligned} \frac{\partial V_{xz1}^j}{\partial z} &= -\tau_{zx} & \frac{\partial V_{xz3}^j}{\partial z} &= \tau_{zx} \\ V_{xz} &= \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} + \frac{\partial V_{xz}^j}{\partial z} & V_{yz} &= \frac{\partial M_y}{\partial x} + \frac{\partial M_{xy}}{\partial y} + \frac{\partial V_{yz}^j}{\partial z} \\ \frac{\partial V_{xz}^j}{\partial z} &= \tau_{xz} h_2 & \frac{\partial V_{yz}^j}{\partial z} &= \tau_{yz} h_2 \end{aligned}$$

$j$  means the sandwich core.

We can choose from typical boundary conditions for rectangular plate:

$$u_i = 0, \quad v_i = 0, \quad w = 0, \quad \frac{\partial w}{\partial n} = 0, \quad M_n = 0, \quad V_n = 0$$

The governing equations are solved by using FEM in program COSMOS/M [3].

### 3. The design optimization problem

An optimization problem has an objective function which measures the goodness or efficiency of design. An optimization process is performed within some limits that constrain the choice of design. Finally, an optimization problem has design variables that are changed during the design process [5].

The basic problem is the minimization of a function subject to inequality constraints.

Minimize objective function:

$$F(\mathbf{X}) \rightarrow \min$$

Subject to:

- side constraints

$$\bar{X}_i^L \leq X_i \leq \bar{X}_i^U \quad i = 1, 2, \dots, N_d$$

- behavior constraints

$$g_j(\mathbf{X}) \leq 0 \quad j = 1, 2, \dots, N_c$$

where:

$$X_i = i^{th} \text{ design variable.}$$

### 3.1. Function Approximation

The main idea of optimization design is based on finding a mathematical relationship between the objective function or constraints and design variables. We make use of the existing response at a number of points in the design space to construct a polynomial approximation to the response at other points [4]. The optimization process is applied to the approximate problem represented by the polynomial approximation.

We use the polynomial approximation:

$$F = a_0 + \sum_{i=1}^{N_d} a_i X_i + \sum_{i=1}^{N_d} b_i X_i^2 + \sum_{i=1}^{N_d-1} \sum_{j=i+1}^{N_d} c_{ij} X_i X_j + \sum_{i=1}^{N_d} d_i X_i^3$$

where:

$N_d$  = number of design variables

$a_i, b_i, c_{ij}, d_i$  = coefficients to be determined

We start with linear approximation and add another terms, if it is needed. For example, if we want to fit response by the linear approximation, we need  $N_d + 2$  design sets to start with a linear approximation. When the optimization loop number exceeds this value, new design sets are added to the linear approximation until it reaches  $2N_d + 2$ . After then, a quadratic approximation is adopted.

The coefficients of the polynomial function are determined by a least squares regression. For regression analysis is used the singular value decomposition (SVD).

### 3.2. The Modified Feasible Direction Method

After the objective function and constraints are approximated and their gradients with respect to the design variables are calculated based on chosen approximation, it is possible to solve the approximate optimization problem. Using the modified feasible direction method (MFD) the solving process is iterated until convergence is achieved. The Fig. 2 [3] shows algorithm of the modified feasible direction method.

In order to make any further improvement in an optimization loop, a new search direction must be found that leads to reduction of the objective function keeping the design feasible.

### 3.3. The Sequential Linear Programming Method

The other algorithm for solving the approximate optimization problem is the sequential linear programming method (SLP). SLP is the iterative process within each optimization loop. The algorithm of SLP is shown in Fig. 3 [3].

### 3.4. Convergence criteria

The several criteria to decide when to end the iterative search process are:

- maximum iterations,
- changes of objective function.

Besides these criteria, the Kuhn-Tucker conditions necessary for optimality must be satisfied.

Another convergence criteria are performed at the end of each optimization loop.

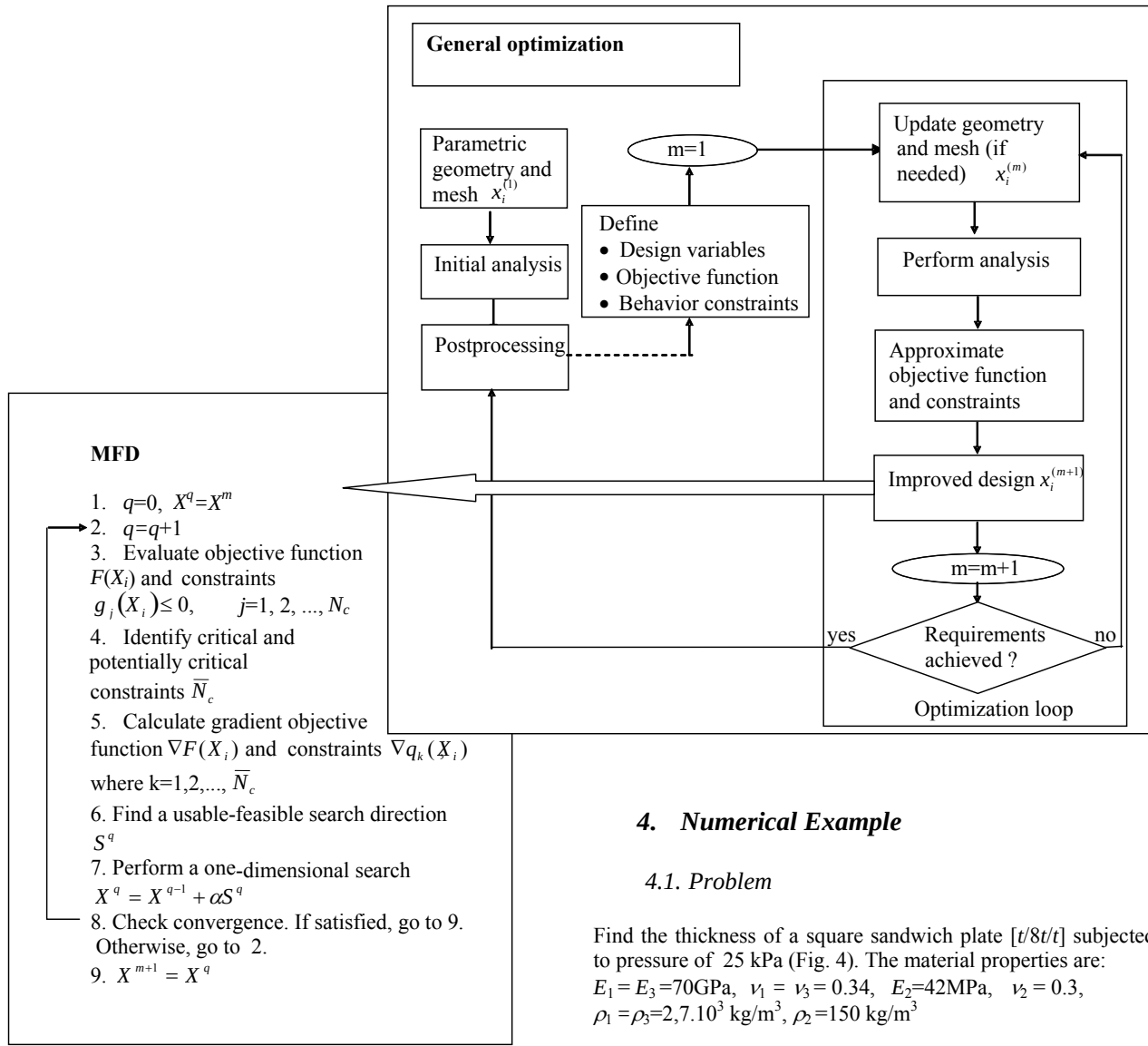


Fig. 2: The modified feasible direction method

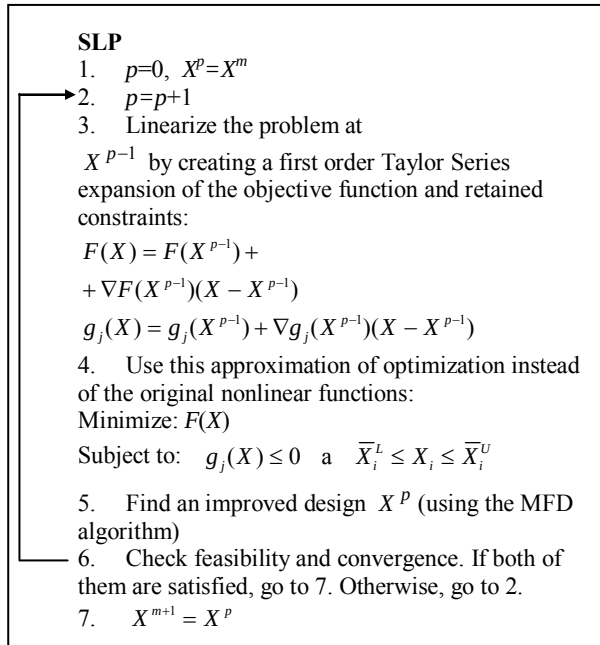


Fig. 3: The sequential linear programming method

## 4. Numerical Example

### 4.1. Problem

Find the thickness of a square sandwich plate  $[t/8t/t]$  subjected to pressure of 25 kPa (Fig. 4). The material properties are:  $E_1 = E_3 = 70 \text{ GPa}$ ,  $\nu_1 = \nu_3 = 0.34$ ,  $E_2 = 42 \text{ MPa}$ ,  $\nu_2 = 0.3$ ,  $\rho_1 = \rho_3 = 2,7 \cdot 10^3 \text{ kg/m}^3$ ,  $\rho_2 = 150 \text{ kg/m}^3$

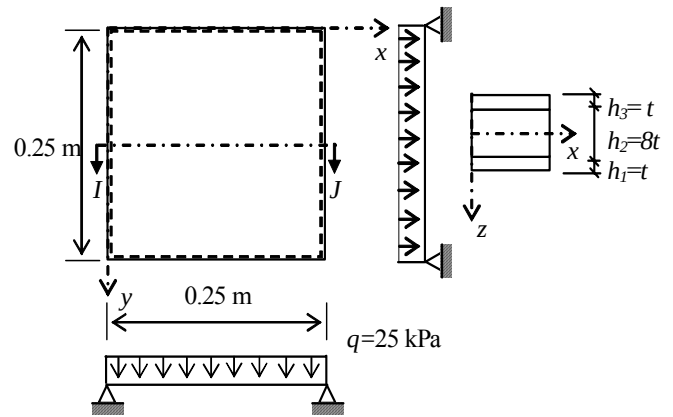


Fig. 4: Problem geometry

The design optimization problem we can write:

$$F(\mathbf{X}) = \sigma^{VM}(t) \rightarrow \min \text{ [MPa]}$$

$$0.0001 \leq t \leq 0.01 \text{ [m]}$$

$$0.41 \leq G(t) \leq 15.1 \text{ [N]}$$

The initial values and bounds of design variables, constraints and the objective function are shown in the Table 1.

## 4.2. Summary of Results

Tab. 1: Optimization parameters

| Optimization parameters |                        | Initial values and bounds      | Final values          | Tolerance         |
|-------------------------|------------------------|--------------------------------|-----------------------|-------------------|
| Design variables        | $t$ [m]                | $0.0001 \leq 0.0005 \leq 0.01$ | $3.669 \cdot 10^{-3}$ | $1 \cdot 10^{-5}$ |
| Objective function      | $\sigma^{VM}(t)$ [MPa] | 37.593                         | 0.7573                | 0.0376            |
| Constraints             | $G(t)$ [N]             | $0.41 \leq 2.063 \leq 15.1$    | 15.1363               | 0.04              |

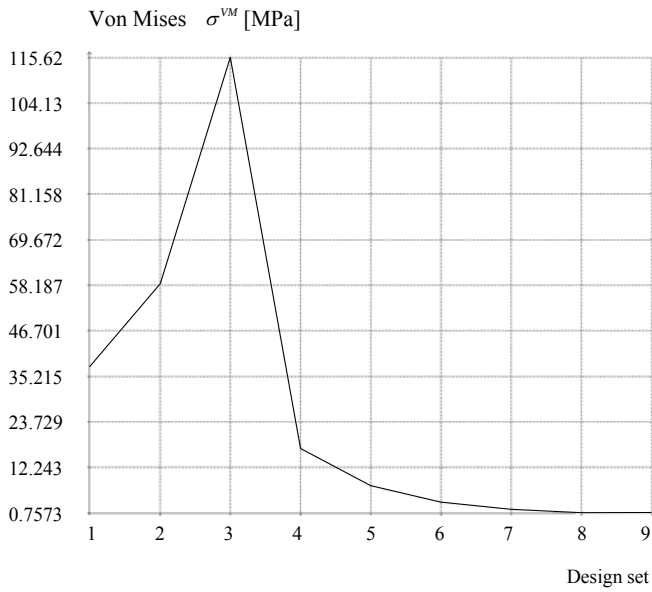


Fig. 5: Variation of the objective function values during the optimization process



Fig. 6: Variation of the design variable values during the optimization process

Von Mises [MPa]

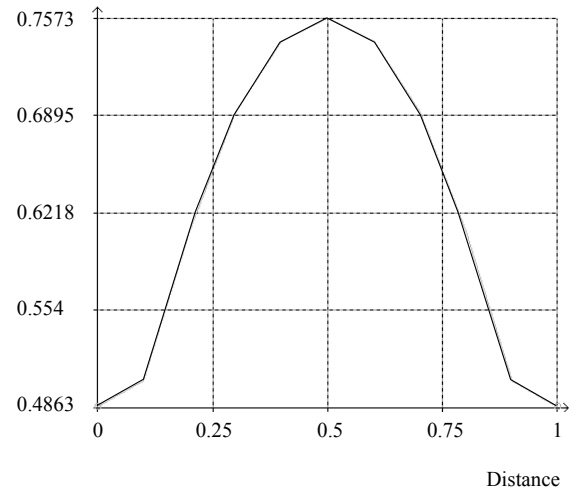


Fig. 7: Von Mises stresses on the sandwich plate at the bottom of the first layer after optimization process – the middle section I-J

## 5. Conclusion

In this paper there are described design optimization problem for sandwich plates. Design of sandwich plates requires both a thorough understanding of the mechanics of sandwich composites and familiarity with optimization techniques that enable designers to find the best design [6]. The paper combines the description of the modelling of sandwich plates with optimization methods that are most useful for the design of such structures.

## 6. References

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